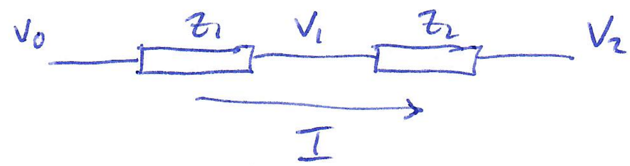
ME 4555 - Lecture 14 - Impulse response

Caution : circuits vs block diagrams. They are different!

circuit :

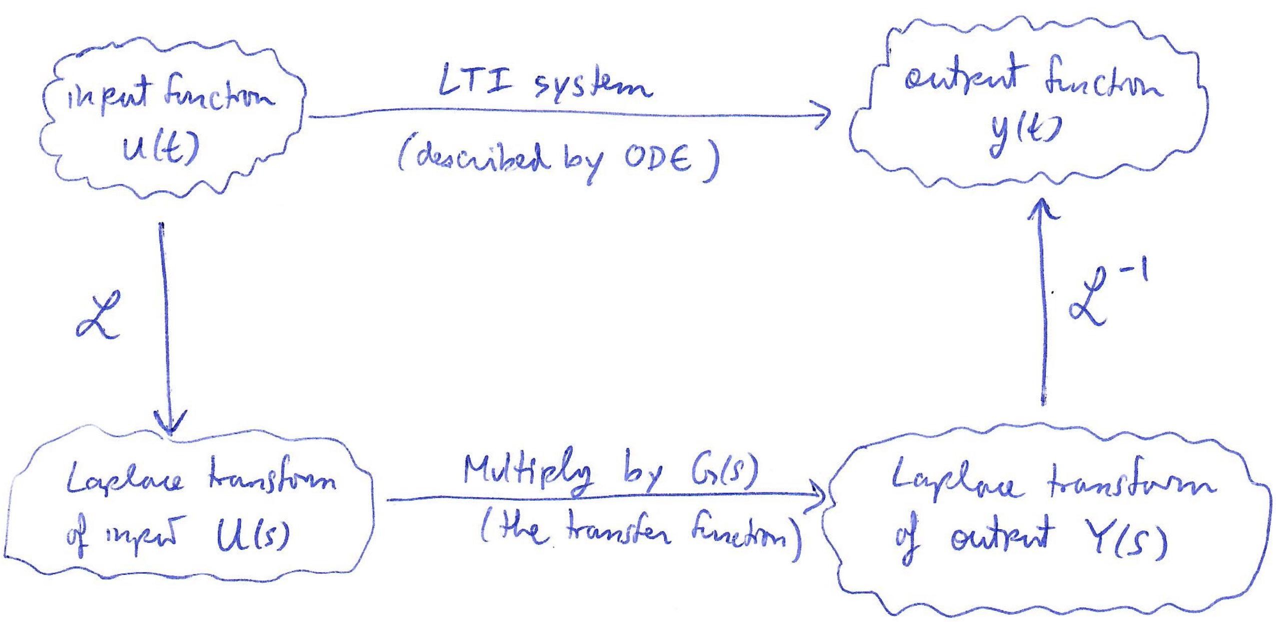


Z_i's are impedances: $V_0 - V_1 = Z_1 I$, $V_1 - V_2 = Z_2 I$ (Ohm's law)

block diagram : $V_0 \longrightarrow \boxed{G_1} \xrightarrow{V_1} \boxed{G_2} \longrightarrow V_2$.

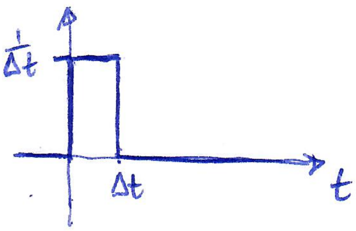
G_i's are transfer functions $\dfrac{V_1}{V_0} = G_1$, $\dfrac{V_2}{V_1} = G_2$

So far, we've seen the following process:



How do we go directly from $u(t)$ to $y(t)$?

Let's construct a new input, the pulse:



has width Δt and height $\frac{1}{\Delta t}$.

Let's call this $\delta_{\Delta t}(t)$.

Construct its Laplace transform:

$$\mathcal{L}\{\delta_{\Delta t}(t)\} = \int_0^\infty \delta_{\Delta t}(t)\, e^{-st}\, dt = \int_0^{\Delta t} \frac{1}{\Delta t} e^{-st}\, dt = \left[-\frac{1}{s\Delta t} e^{-st}\right]_0^{\Delta t}$$

$$\Rightarrow \mathcal{L}\{\delta_{\Delta t}(t)\} = \frac{1}{s\Delta t}\left(1 - e^{-s\Delta t}\right)$$

use Taylor Series expansion: $e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$

$$\Rightarrow \mathcal{L}\{\delta_{\Delta t}(t)\} \approx 1 \underbrace{- \frac{1}{2} s\Delta t + \dots}_{\text{higher order terms in } \Delta t}$$

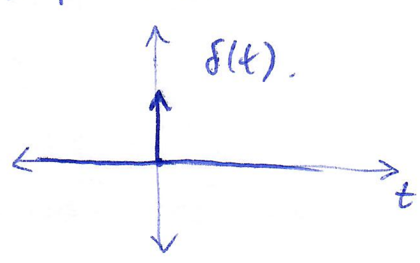
We define a new test input, which can be thought of as the limit $\Delta t \to 0$ of $\delta_{\Delta t}(t)$. It has the following properties:

$$\boxed{\begin{gathered} \delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \\ \mathcal{L}\{\delta(t)\} = 1. \end{gathered}}$$

This is called the Dirac Delta function.

Note: it's not really a "function"; strictly speaking it is a distribution.

We graph it as:



☆ if we use $\delta(t)$ as an input to a system, the output is called the $\boxed{\text{impulse response}}$

☆ the transfer function is the laplace transform of the impulse response! To see why, note that:

$$\underset{\substack{\uparrow \\ \text{TF}}}{G(s)} = \frac{Y(s)}{\underset{\substack{\uparrow \\ \text{input}}}{U(s)}} \leftarrow \text{output.}$$

if $u(t) = \delta(t)$, then

$$U(s) = \mathcal{L}\{\delta(t)\} = 1.$$

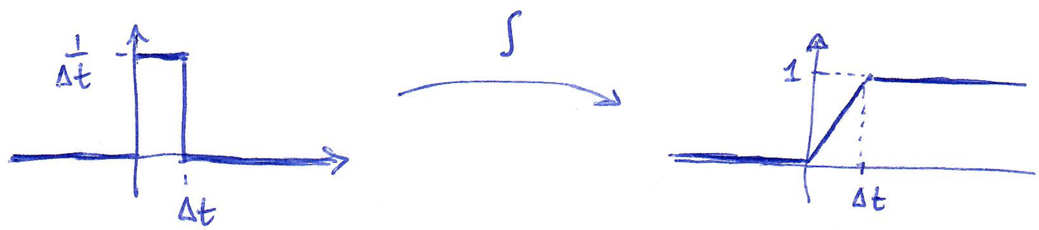
so $G(s) = Y(s)$.

So if we use $\delta(t)$ as input and observe $\underbrace{g(t)}_{\text{(impulse response)}}$ as output

then $G(s) = \mathcal{L}\{g(t)\}$.

☆ The integral of $\delta(t)$ is $H(t)$ (Heaviside step function).

Note that: $\displaystyle\int_{-\infty}^{t} \delta_{\Delta t}(\tau)\, d\tau$



So in the limit, $\delta(t) \xrightarrow{\int} H(t)$.

This explains why in s-domain,

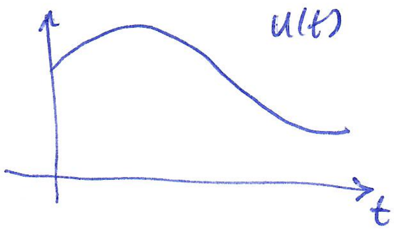
$$\underset{\substack{\uparrow \\ \mathcal{L}\{\delta(t)\}}}{1} \longrightarrow \underset{\substack{\uparrow \\ \text{TF for integrator}}}{\boxed{\tfrac{1}{s}}} \longrightarrow \underset{\substack{\uparrow \\ \mathcal{L}\{H(t)\}.}}{\frac{1}{s}}$$

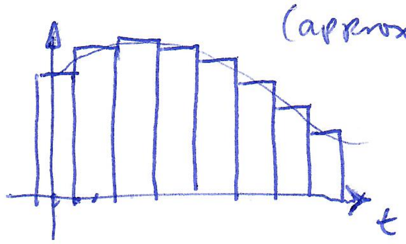
Also why we use symbol $\boxed{\tfrac{1}{s}}$ in Simulink for integration!

if pulse input $\delta_{\Delta t}(t)$ has output $g_{\Delta t}(t)$

then the limit delta function $\delta(t)$ has output $g(t)$ (impulse response).

Take a general input $u(t)$ and approximate it as a sum of shifted pulses, then apply the superposition principle.





(approximation)

limit as $\Delta t \to 0$

$$u(t) \approx \sum_{i=0}^{\infty} \underbrace{u(i\Delta t)}_{\text{discretized input}} \underbrace{\delta_{\Delta t}(t-i\Delta t)\,\Delta t}_{\text{shifted pulse of height 1}}$$

$$u(t) = \int_0^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$$

apply LTI system

use homogeneity and superposition

apply LTI system

$$y(t) \approx \sum_{i=0}^{\infty} u(i\Delta t)\, g_{\Delta t}(t-i\Delta t)\,\Delta t$$

$$y(t) = \int_0^{\infty} u(\tau)\, g(t-\tau)\, d\tau$$

we can show $\int_0^{\infty} u(\tau) g(t-\tau)\,d\tau = \int_0^{\infty} g(\tau)\, u(t-\tau)\,d\tau$.

Often written $(g * u)(t)$. This is called the <u>convolution</u> of g and u

<u>Summary</u>: If $g(t)$ is the impulse response, $G(s) = \mathcal{L}\{g(t)\}$ is TF:

$$\begin{tikzcd}
u(t) \arrow[rr, "\text{apply LTI system}"] \arrow[d, "\mathcal{L}"'] & & y(t) = (g * u)(t) \\
U(s) \arrow[rr, "\text{transfer function}"] & & Y(s) = G(s)\,U(s) \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

So <u>convolution</u> in the time domain is <u>multiplication</u> in the s-domain.

$$\mathcal{L}\{f * h\} = \mathcal{L}\{f\}\,\mathcal{L}\{h\}.$$

or in other words, $\mathcal{L}\{(f * h)(t)\} = F(s)\,H(s)$

$f(t)$	$F(s)$		
$\delta(t)$	1	← impulse response	
$H(t)$	$\frac{1}{s}$	← step response	test inputs
$H(t)\cdot t$	$\frac{1}{s^2}$	← ramp input	
af	aF	← homogeneity	linearity
$f+g$	$F+G$	← superposition	
$f*g$	FG	← convolution	